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Generalized Niederer's transformation for quantum Pais–Uhlenbeck oscillator

Krzysztof Andrzejewski

Department of Computer Science, University of Łódź, Pomorska 149/153, 90-236 Łódź, Poland

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Abstract

We extend, to the quantum domain, the results obtained in [Nucl. Phys. B 885 (2014) 150] and [Phys. Lett. B 738 (2014) 405] concerning Niederer's transformation for the Pais–Uhlenbeck oscillator. Namely, the quantum counterpart (an unitary operator) of the transformation which maps the free higher derivatives theory into the Pais–Uhlenbeck oscillator is constructed. Some consequences of this transformation are discussed.

1. Introduction

It is well known that the harmonic oscillator motion can be mapped into the free one. More precisely, the following point transformation[1]

$$t = t(\tilde{t}) \equiv \frac{1}{\omega} \arctan(\omega \tilde{t}), \quad Q = Q(\tilde{q}, \tilde{t}) \equiv \frac{\tilde{q}}{\tilde{\kappa}}, \tag{1.1}$$

where

$$\tilde{\kappa} = \tilde{\kappa}(\tilde{t}) \equiv \sqrt{1 + \omega^2 \tilde{t}^2}, \tag{1.2}$$

E-mail address: k-andrzejewski@uni.lodz.pl.

[1] To simplify the notation we omit the spatial indices and put mass equal to one. Throughout this paper the tilde sign refers to the free case.

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connects the classical free motion (described by the $\tilde{q}$ coordinate and $-\infty < \tilde{t} < \infty$) with half of the period motion of the harmonic oscillator (described by the Q coordinate and $-\frac{\pi}{2\omega} < t < \frac{\pi}{2\omega}$), i.e. the following identity holds

$$\left[\frac{1}{2}\left(\frac{dQ}{dt}\right)^2 - \frac{\omega^2 Q^2}{2}\right] dt = \frac{1}{2}\left(\frac{d\tilde{q}}{d\tilde{t}}\right)^2 d\tilde{t} - d\left(\frac{\tilde{t}\omega^2\tilde{q}^2}{2\tilde{\kappa}^2}\right). \tag{1.3}$$

In the other words, eq. (1.3) tells us that $\tilde{q}$ describes the free motion provided Q obeys the harmonic oscillator equation of motion (and vice versa).

What is more this transformation has a counterpart, obtained by Niederer [1], in quantum mechanics. Namely, if $\phi(Q,t)$ obeys the Schrödinger equation for the harmonic oscillator then

$$\begin{aligned}\chi(\tilde{q},\tilde{t}) &= \tilde{\kappa}^{-\frac{1}{2}} e^{i\frac{\tilde{t}\omega^2\tilde{q}^2}{2\tilde{\kappa}^2}} \psi(Q(\tilde{q},\tilde{t}), t(\tilde{t})) \\ &\equiv \tilde{\kappa}^{-\frac{1}{2}} e^{i\frac{\tilde{t}\omega^2\tilde{q}^2}{2\tilde{\kappa}^2}} \psi(\frac{\tilde{q}}{\tilde{\kappa}}, \frac{1}{\omega}\arctan(\omega\tilde{t})),\end{aligned} \tag{1.4}$$

is a solution to the free Schrödinger equation. Moreover, as one expected, the phase factor in the transformation (1.4) is exactly equal to the function which enters into the total time derivative relating both Lagrangians, cf. eq. (1.3).

Of course, the above observation does not mean that the classical (quantum) free motion is equivalent to the harmonic one due to the fact that the transformation does not have a global form. However, such information reflects a similarity between the both systems and offers simpler explanation of some facts. For example, it implies that for the both systems the maximal symmetry groups (algebras) are isomorphic to each other (on the classical level to the Schrödinger algebra and to its central extension on the quantum level). Moreover, we can transform various quantities (symmetry generators, Feynman's propagator, etc.) from one system to the other system (see e.g. [1–5]); in particular, we have the explicit relation between their solutions which enables us their better analysis [6,7]. Finally, such mapping is an important example of the Arnold transformation [8,9] and appears in the context of the nonlinear Schrödinger equation with more complicated potentials [10,11].

On the other hand, we observe the increasing interest in theories containing higher order derivatives. Originally, these theories were proposed as a method for dealing with ultraviolet divergences. This idea was briefly mentioned in Ref. [12] and next fully developed in Ref. [13]. Similar idea of adding higher derivative terms was also proposed as a method of regularizing Einstein gravity by supplying the Einstein action with the terms containing higher powers of the curvature which lead to a renormalizable theory [14]. However, it should be noted that the original Einstein theory is not renormalizable so adding such terms becomes an essential modification of the theory making it renormalizable at the price of dealing with ghosts. Other examples of higher derivatives theories include the theory of the radiation reaction [15,16], the field theory on noncommutative spacetime [17,18], anyons [19,20] or string theories with the extrinsic curvature [21].

The simplest theory with higher time derivatives is the one defined by the following Lagrangian (generalization of the ordinary free motion, $n = 1$)

$$\tilde{L} = \frac{(-1)^{n-1}}{2}\left(\frac{d^n\tilde{q}}{d\tilde{t}^n}\right)^2, \quad n = 1, 2, \ldots \tag{1.5}$$

The dynamical equation takes the form

$$\frac{d^{2n}\tilde{q}}{d\tilde{t}^{2n}} = 0. \tag{1.6}$$

There exists also the generalization of harmonic oscillator to the case of higher derivatives. Such system was proposed by Pais and Uhlenbeck (PU) in their classical paper [22] and is defined by the Lagrangian

$$L = -\frac{1}{2} Q \prod_{k=1}^{n} \left(\frac{d^2}{dt^2} + \omega_k^2 \right) Q, \tag{1.7}$$

where $0 < \omega_1 < \omega_2 < \ldots < \omega_n$ and $n = 1, 2, \ldots$. Lagrangian (1.7) implies the following equation of motion

$$\prod_{k=1}^{n} \left(\frac{d^2}{dt^2} + \omega_k^2 \right) Q = 0. \tag{1.8}$$

Since Lagrangian (1.7) is linear in the highest derivative (and thus singular) it is advantageous to expand it in the sum of higher derivatives terms and next integrate by parts. As a consequence we arrive at the following, equivalent, Lagrangian which is nonsingular and also called PU oscillator Lagrangian

$$L = \frac{1}{2} \sum_{k=0}^{n} (-1)^{k-1} \sigma_k \left(\frac{d^k Q}{dt^k} \right)^2, \tag{1.9}$$

where

$$\sigma_k = \sum_{i_1 < \ldots < i_{n-k}} \omega_{i_1}^2 \cdots \omega_{i_{n-k}}^2, \quad k = 0, \ldots, n; \quad \sigma_n = 1. \tag{1.10}$$

The PU model has been attracting considerable interest throughout the years (for the last few years, see e.g. [23–41]) and it can serve to achieve a deeper insight into problems (and their solutions) which emerge for more complicated higher derivatives theories.

In the context of Niederer's results (disused above) the following question arises: whether the free higher derivatives theory can be related to the PU model and what are the reasons and consequences of the existence of such relation. Surprisingly enough, in this case the situation is more involved. On the classical level it was shown [41] that only for *odd* frequencies, i.e. when they form an arithmetic sequence, $\omega_k = (2k-1)\omega, \quad \omega \neq 0$, for $k = 1, \ldots, n$, the PU oscillator can be related, by a generalization of Niederer's transformation, to the free higher derivatives motion. More precisely, the following transformation

$$t = \frac{1}{\omega} \arctan(\omega \tilde{t}), \quad Q = \frac{1}{\tilde{\kappa}^{2n-1}} \tilde{q}, \tag{1.11}$$

(which was suggested in Ref. [42]) transforms eq. (1.8) (with odd frequencies) into eq. (1.6) and consequently establishes the desired relation (for the Lagrangian (1.7) see also [43]). Moreover, it was shown in Ref. [41] that, for such frequencies, on the classical level the maximal symmetry group of the PU oscillator is isomorphic to the maximal symmetry group of the free higher derivatives theory, i.e. to the l-conformal Galilei group (with $l = n - \frac{1}{2}$, see [44]). For other frequencies the symmetry group has simpler form (the are no conformal and dilatation transformations). Therefore, we can expect that only in the case of odd frequencies the PU oscillator can be related to the free higher derivatives motion.

Much attention has been also paid to the Hamiltonian formulations of the PU oscillator. There exist a few approaches: decomposition into the set of independent harmonic oscillators proposed by Pais and Uhlenbeck in their original paper [22], Ostrogradsky approach based on the Ostrogradsky method [45] of constructing Hamiltonian formalism for theories with higher time derivatives and the one (see [46]), applicable in the case of odd frequencies (mentioned above), which exhibits the conformal group structure of the model; the latter will be called *algebraic approach* since the Hamiltonian is built (in analogy with the ordinary oscillator) out of the Hamiltonian and the conformal generator of the free theory.

In this paper we complete the picture and show that on the quantum level the free higher derivatives theory is related, by an unitary transformation, to PU oscillator. Moreover, the phase factor appearing in this transformation coincides with the total time derivative on the Lagrangian level; however, we must take the Lagrangians quadratic in velocities, see (1.5) and (1.9). These results imply the form of the symmetry group of the quantum PU oscillator with odd frequencies.

The paper is organized as follows. In Section 2 we remind briefly the results obtained in Ref. [46] concerning various Hamiltonian approaches for the PU model and we derive some new relations needed in what follows. In Section 3 we construct an unitary operator which maps the Schrödinger equation for the PU oscillator into the free Schrödinger equation. Section 4 shows that the well known relation between the phase factor in the unitary operator and total time derivative on the Lagrangian level holds also in our case if we take the appropriate Lagrangians. Finally, in Section 5 we summarize our results and discuss possible further developments.

2. Hamiltonian formalisms for PU model

In this section we recall three main Hamiltonian formalisms for the PU model and relations between them; we derive also some relations used in the next sections. The first Hamiltonian formalism (proposed in Ref. [22]) is based on decoupled oscillators where the Hamiltonian is the sum of harmonic Hamiltonians with alternating sign. In this approach we introduce new variables

$$x_k = \Pi_k Q, \quad k = 1, \dots, n; \tag{2.1}$$

where Π_k is the projector operator:

$$\Pi_k = \sqrt{|\rho_k|} \prod_{\substack{i=1 \\ i \neq k}}^{n} \left(\frac{d^2}{dt^2} + \omega_i^2 \right), \tag{2.2}$$

and

$$\rho_k = \frac{1}{\prod_{\substack{i=1 \\ i \neq k}}^{n} (\omega_i^2 - \omega_k^2)}, \quad k = 1, 2, \dots, n. \tag{2.3}$$

Note that ρ_k are alternating in sign and, in the case of odd frequencies ($\omega_k = (2k-1)\omega$), they have the following explicit form

$$\rho_k = \frac{(-1)^{k-1}(2k-1)}{(4\omega^2)^{n-1}(n-k)!(n+k-1)!}, \quad k = 1, \dots, n. \tag{2.4}$$

Next, one finds

$$L = -\frac{1}{2}\sum_{k=1}^{n}(-1)^{k-1}x_k\left(\frac{d^2}{dt^2}+\omega_k^2\right)x_k = \frac{1}{2}\sum_{k=1}^{n}(-1)^{k-1}(\dot{x_k}^2-\omega_k^2x_k^2)+t.d. \tag{2.5}$$

The corresponding Hamiltonian reads

$$H_1 = \frac{1}{2}\sum_{k=1}^{n}(-1)^{k-1}(p_k^2+\omega_k^2x_k^2). \tag{2.6}$$

The second Hamiltonian formalism is obtained by the method proposed by Ostrogradsky [45] for Lagrangian with higher derivatives. To this end we need the Lagrangian which is nonsingular in the highest derivative; in our case it is given by eq. (1.9). Next, we introduce the Ostrogradsky variables

$$\begin{aligned} Q_k &= Q^{(k-1)}, \\ \Pi_k &= \sum_{j=0}^{n-k}\left(-\frac{d}{dt}\right)^j\frac{\partial L}{\partial Q^{(k+j)}} = (-1)^{k-1}\sum_{j=k}^{n}\sigma_j Q^{(2j-k)}, \end{aligned} \tag{2.7}$$

for $k = 1, \dots, n$. Then the Ostrogradsky Hamiltonian takes the form

$$H_2 = \frac{(-1)^{n-1}}{2}\Pi_n^2 + \sum_{k=2}^{n}\Pi_{k-1}Q_k - \frac{1}{2}\sum_{k=1}^{n}(-1)^k\sigma_{k-1}Q_k^2. \tag{2.8}$$

Finally, for odd frequencies we have an additional form of Hamiltonian formalism [46]. It is based on the observation that, as in the case of ordinary harmonic oscillator, the Hamiltonian can be written as the sum of the free Hamiltonian and the conformal generator of the free theory. In consequence we obtain

$$\begin{aligned} H_3 = {} & \frac{(-1)^{n+1}}{2}\pi_{n-1}^2 - \sum_{m=1}^{n-1}q_m\pi_{m-1} + (-1)^{n+1}\frac{n^2\omega^2}{2}q_{n-1}^2 \\ & + \sum_{m=0}^{n-2}(2n-1-m)(m+1)\omega^2 q_m\pi_{m+1}. \end{aligned} \tag{2.9}$$

In this approach the variables $q_m, \pi_m, m = 0, \dots, n-1$ correspond to the Ostrogradsky variables of the free theory. The relations between these approaches (i.e. the canonical transformations which relate to each other) are described in Ref. [46] from which we adopt the notation. The Ostrogradsky approach and the one base on decoupled oscillators are related by the following canonical transformation

$$\begin{aligned} x_i &= {\sum_{k=1}^{n}}'(-1)^{\frac{k-3}{2}}\sum_{j=k}^{n}\sigma_j(-1)^j\omega_i^{2j-k-1}\sqrt{|\rho_i|}Q_k + {\sum_{k=1}^{n}}''(-1)^{\frac{k}{2}}\sqrt{|\rho_i|}\omega_i^{k-2}\Pi_k, \\ p_i &= {\sum_{k=1}^{n}}''(-1)^{\frac{k}{2}+i-1}\sum_{j=k}^{n}\sigma_j(-1)^j\omega_i^{2j-k}\sqrt{|\rho_i|}Q_k + {\sum_{k=1}^{n}}'(-1)^{\frac{k+1}{2}+i}\sqrt{|\rho_i|}\omega_i^{k-1}\Pi_k; \end{aligned} \tag{2.10}$$

while (for odd frequencies) one can pass from the algebraic approach to the decoupled oscillators by the canonical transformation (see Appendix)

$$
\begin{aligned}
x_k &= (-1)^k \left(\sum_{m=0}^{n-1}{}'' \frac{\omega^{-m}}{m!\sqrt{|\rho_k|}} \gamma^+_{km} q_m + \sum_{m=0}^{n-1}{}' \frac{m!\omega^m \sqrt{|\rho_k|}}{(2k-1)\omega} \beta^+_{2n-1-m,k} \pi_m \right), \\
p_k &= (-1)^k \left(-\sum_{m=0}^{n-1}{}' \frac{\omega^{-m}(2k-1)\omega}{m!\sqrt{|\rho_k|}} \gamma^+_{k,2n-1-m} q_m + \sum_{m=0}^{n-1}{}'' m!\omega^m \sqrt{|\rho_k|} \beta^+_{mk} \pi_m \right),
\end{aligned}
\tag{2.11}
$$

for $k = 1, \dots, n$; one and two primes denote the sum over odd and even indices, respectively (we corrected here a misprint in sign in Ref. [46]).

One can observe (using the inverse of Vandermonde matrix) that the transformation (2.10) is the composition of a canonical point transformation and the partial exchange of coordinates and momenta

$$
(Q_k, \Pi_k) \to (Q_k, \Pi_k), \quad k\text{-odd}; \qquad (Q_k, \Pi_k) \to (-\Pi_k, Q_k) \quad k\text{-even}. \tag{2.12}
$$

The same holds true in the case of the transformation (2.11), but this time one has

$$
(q_m, \pi_m) \to (q_m, \pi_m), \quad m\text{-even}; \qquad (q_k, \pi_m) \to (-\pi_m, q_m) \quad m\text{-odd}. \tag{2.13}
$$

The inverse transformation to (2.10) takes the form

$$
\begin{aligned}
Q_k &= (-1)^{\frac{k-1}{2}} \sum_{j=1}^{n} \sqrt{|\rho_j|}(-1)^{j-1} \omega_j^{k-1} x_j, \quad k\text{-odd}; \\
Q_k &= (-1)^{\frac{k}{2}-1} \sum_{j=1}^{n} \sqrt{|\rho_j|} \omega_j^{k-2} p_j, \quad k\text{-even};
\end{aligned}
\tag{2.14}
$$

and

$$
\begin{aligned}
\Pi_k &= (-1)^{\frac{k}{2}-1} \sum_{i=1}^{n} (-1)^{i-1} \sqrt{|\rho_i|} \left(\sum_{j=k}^{n} \sigma_j (-1)^j \omega_i^{2j-k} \right) x_i, \quad k\text{-even}; \\
\Pi_k &= (-1)^{\frac{k-3}{2}} \sum_{i=1}^{n} \sqrt{|\rho_i|} \left(\sum_{j=k}^{n} \sigma_j (-1)^j \omega_i^{2j-k-1} \right) p_i, \quad k\text{-odd}.
\end{aligned}
\tag{2.15}
$$

for $k = 1, \dots, n$.

Using (2.11), (2.14) and (2.15) we can find the canonical transformation leading from (q_m, π_m) to (Q_k, P_k) variables. Indeed, after some computations, using (A.2)–(A.7), we obtain the following general[2] point transformation

$$
Q_k = \sum_{m=0}^{n-1}{}'' X^+_{km} q_m, \quad k\text{-odd}, \qquad Q_k = \sum_{m=0}^{n-1}{}' X^-_{km} q_m, \quad k\text{-even}; \tag{2.16}
$$

$$
\begin{aligned}
\Pi_k &= \sum_{\bar{m}=0}^{n-1}{}'' ((X^+)^{-1})_{\bar{m}k} \left(\pi_{\bar{m}} + \sum_{m=0}^{n-1}{}' Y^1_{m\bar{m}} q_m \right), \quad k\text{-odd}; \\
\Pi_k &= \sum_{m=0}^{n-1}{}' ((X^-)^{-1})_{mk} \left(\pi_m + \sum_{\bar{m}=0}^{n-1}{}'' Y^2_{\bar{m}m} q_{\bar{m}} \right), \quad k\text{-even};
\end{aligned}
\tag{2.17}
$$

[2] It differs from the ordinary canonical point transformation by the additional terms in q in the expression for momenta.

where

$$X^{\pm}_{km} = (-1)^{[\frac{k}{2}]+1}\frac{\omega^{-m}}{m!}\sum_{r=1}^{n}\gamma^{\pm}_{rm}\omega_r^{k-1}, \tag{2.18}$$

and the matrices $Y^{1,2}$ are of the form

$$Y^1_{m\bar{m}} = \frac{\omega^{-m-\bar{m}}}{m!\bar{m}!}\sum_{j=1}^{n}\sigma_j(-1)^j\sum_{k,\bar{k}=1}^{n}\gamma^{-}_{km}\gamma^{+}_{\bar{k}\bar{m}}\sum_{r=1}^{j}{}' \omega_k^{2j-r}\omega_{\bar{k}}^{r-1}, \tag{2.19}$$

$$Y^2_{\bar{m}m} = -\frac{\omega^{-m-\bar{m}}}{m!\bar{m}!}\sum_{j=1}^{n}\sigma_j(-1)^j\sum_{k,\bar{k}=1}^{n}\gamma^{-}_{km}\gamma^{+}_{\bar{k}\bar{m}}\sum_{r=1}^{j}{}'' \omega_{\bar{k}}^{2j-r}\omega_{k}^{r-1}, \tag{2.20}$$

for m odd and $\bar{m}$ even. Moreover, using (A.2) and (A.3) one can show that

$$Y^1_{m\bar{m}} = Y^2_{\bar{m}m}, \tag{2.21}$$

for $m, \bar{m} = 0, \dots, n-1$, m-odd and $\bar{m}$-even. The generating function of the transformation (2.16) and (2.17) is of the form

$$F(q_0,\dots,q_{n-1},\Pi_1,\dots,\Pi_n) = \sum_{k=1}^{n}\Pi_k Q_k(q_0,\dots,q_{n-1}) + f(q_0,\dots,q_{n-1}), \tag{2.22}$$

where $Q_k(q_0,\dots,q_{n-1})$ are given by eq. (2.16) and the function f reads

$$f(q_0,\dots,q_{n-1}) = -\sum_{m=0}^{n-1}{}'\sum_{\bar{m}=0}^{n-1}{}'' Y^1_{m\bar{m}}q_m q_{\bar{m}}. \tag{2.23}$$

3. Quantum Niederer's transformation for PU model

In this section we construct the quantum version of Niederer's transformation for PU oscillator. In order to do this we need the canonical transformation, constructed in Ref. [47], relating the Hamiltonian $\tilde{H}$ of the free theory

$$\tilde{H} = \frac{(-1)^{n+1}}{2}\tilde{\pi}^2_{n-1} + \sum_{k=1}^{n-1}\tilde{\pi}_{k-1}\tilde{q}_k, \tag{3.1}$$

to the PU Hamiltonian H_3 (in the algebraic approach). Adapting to our conventions the results of Ref. [47] and performing some manipulations we obtain the following transformation

$$q_k = \sum_{m=0}^{n-1}B_{km}\tilde{q}_m, \tag{3.2}$$

$$\pi_k = \sum_{m=0}^{n-1}(B^{-1})_{mk}(\tilde{\pi}_m + \sum_{j=0}^{n-1}C_{jm}\tilde{q}_j), \tag{3.3}$$

where

$$B_{km} = (-1)^m \frac{k!}{m!}\binom{2n-1-m}{2n-1-k}\dot{\tilde{\kappa}}^{k-m}\tilde{\kappa}^{m+k-2n+1}, \tag{3.4}$$

$$C_{km} = \frac{(-1)^{n+m+k}}{2n-1-k-m}\frac{(2n-1-k)!}{k!(n-1-k)!}\frac{(2n-1-m)!}{m!(n-1-m)!}\left(\frac{\dot{\tilde{\kappa}}}{\tilde{\kappa}}\right)^{2n-1-k-m}, \tag{3.5}$$

$$(B^{-1})_{km} = \tilde{\kappa}^{2(2n-1-m-k)}B_{km}, \tag{3.6}$$

while, by definition, $\binom{k}{m} = 0$ if $k < m$ and $\dot{\tilde{\kappa}} = \frac{d\tilde{\kappa}}{d\tilde{t}}$. The above transformation yields the identity (see [47])

$$\tilde{H} \equiv H\frac{dt}{d\tilde{t}} + \frac{\partial G}{\partial \tilde{t}}; \tag{3.7}$$

where G is the generating function of the transformation (3.2), (3.3). For our further considerations we take G depending on the old coordinates $\tilde{q}$'s and the new momenta π's. With this choice of the variables it reads

$$G(\tilde{q}_0, \ldots, \tilde{q}_{n-1}, \pi_0, \ldots, \pi_{n-1}, \tilde{t}) = \sum_{k=0}^{n-1} q_k(\tilde{q}_0, \ldots, \tilde{q}_{n-1}, \tilde{t})\pi_k + g(\tilde{q}_0, \ldots, \tilde{q}_{n-1}, \tilde{t}), \tag{3.8}$$

where

$$g(\tilde{q}_0, \ldots, \tilde{q}_{n-1}, \tilde{t}) = -\frac{1}{2}\sum_{k,m=0}^{n-1} C_{km}\tilde{q}_k\tilde{q}_m, \tag{3.9}$$

and $q_k(\tilde{q}_0, \ldots, \tilde{q}_{n-1}, \tilde{t})$ are given by (3.2). Finally, let us compute the Jacobian of the transformation (3.2) or, equivalently, the determinant of the matrix B. Using (3.6) one obtains

$$|\det B| = \tilde{\kappa}^{-n^2}. \tag{3.10}$$

Now, we are ready to construct the quantum version of the transformation (3.2), i.e., an unitary operator which maps the solution $\psi = \psi(q_0, \ldots, q_{n-1}, t)$ of the Schrödinger equation for the PU oscillator in algebraic approach

$$(i\partial_t - \hat{H}_3)\psi = 0, \tag{3.11}$$

to the solution $\chi = \chi(\tilde{q}_0, \ldots, \tilde{q}_{n-1}, \tilde{t})$ of the free Schrödinger equation

$$(i\partial_{\tilde{t}} - \hat{\tilde{H}})\chi = 0, \tag{3.12}$$

where both Hamiltonians are written in the coordinate representation. Taking into account our considerations, we postulate the following form of the unitary operator

$$\begin{aligned}(\hat{U}\psi)(\tilde{q}_0, \ldots, \tilde{q}_{n-1}, \tilde{t}) = \tilde{\kappa}^{-\frac{n^2}{2}} e^{ig(\tilde{q}_0,\ldots,\tilde{q}_{n-1},\tilde{t})}\psi(q_0(\tilde{q}_0, \ldots, \tilde{q}_{n-1}, \tilde{t}),\\ \ldots, q_{n-1}(\tilde{q}_0, \ldots, \tilde{q}_{n-1}, \tilde{t}), t(\tilde{t})),\end{aligned} \tag{3.13}$$

where g, q_m and t are given by (3.9), (3.2) and (1.11) respectively. The structure of the operator $\hat{U}$ is as follows. First, the arguments of the wave function are replaced by the appropriate functions of the new ones according to the classical formulae; then the two factors are added: the first one accounts for proper normalization while the other one is related to the second term in the generating function.

Substituting (3.13) into eq. (3.12) and using the fact that ψ satisfies eq. (3.11) one can check, after some troublesome computations, that $\hat{U}\psi$ satisfies the free Schrödinger equation (3.12). To this end eq. (1.2) and the following identities appear to be useful

$$k(2n-k)\omega^2 B_{k-1,m} - B_{k+1,m} = \tilde{\kappa}^2 B_{k,m-1} + \tilde{\kappa}^2 \frac{\partial B_{km}}{\partial \tilde{t}}, \tag{3.14}$$

$$\frac{\partial C_{km}}{\partial \tilde{t}} + C_{m,k-1} + C_{k,m-1} + (-1)^n C_{m,n-1} C_{k,n-1} = \frac{n^2\omega^2(-1)^n}{\tilde{\kappa}^2} B_{n-1,m} B_{n-1,k}, \tag{3.15}$$

for $m, k = 0, \ldots, n-1$ (B_{km} is also well defined for $k = n$).

Now, the extension of the transformation (3.13) to the remaining two Hamiltonian formalisms is straightforward. Namely, the canonical transformation (2.16)–(2.17) from the algebraic approach to the Ostrogradsky one is a general time-independent point transformation; what is more the structure of this transformation and the Hamiltonians are such that we do not have to care about the ordering (contrary to the previous case) and, therefore, it can be directly defined on the quantum level (see e.g. [48–50]). The only thing we need is the Jacobian of the transformation (2.16). This Jacobian is the product of determinants of the matrices X^+ and X^-; due to the identities (A.5)–(A.7) one can show that its absolute value is equal to one. Consequently, the corresponding unitary operator is of the form

$$(\hat{V}\phi)(q_0, \ldots, q_{n-1}, t) = e^{if(q_0,\ldots,q_{n-1})}\phi(Q_1(q_0, \ldots, q_{n-1}), \ldots, Q_n(q_0, \ldots, q_n), t), \tag{3.16}$$

where $\phi(Q_1, \ldots, Q_n, t)$ is a solution of the Schrödinger equation with the Hamiltonian H_2, f is given by (2.23) and Q_k are expressed by eq. (2.16)

The composition

$$\hat{W} = \hat{U}\hat{V}, \tag{3.17}$$

maps the solutions of the Schrödinger equation for the PU model in Ostrogradsky approach

$$(i\partial_t - \hat{H}_2)\phi = 0, \tag{3.18}$$

into the solutions of the free Schrödinger equation (3.12).

The similar situation appears if we want to pass to the decoupled harmonic oscillators formalism. There is one difference here; as we noted earlier, the canonical transformation (2.11) is the composition of a point one with the partial exchange of coordinates and momenta, cf. eq. (2.13). Therefore, we must additionally perform the Fourier transform in the odd variables.

Finally, let us note that the unitary relation between the free higher derivatives theory and the PU oscillator with the odd frequencies establisher here leads to the conclusion that the maximal quantum (kinematical) symmetry groups for the both systems are isomorphic (this isomorphism is related to different choice of the Hamiltonian as a element of algebra's basis). Since for the higher order quantum free theory the maximal symmetry is the centrally extended l-conformal Galilei algebra ($l = n - \frac{1}{2}$) we obtain the characterization of the symmetry group for the quantum PU model with odd frequencies.

4. The phase factor on the Lagrangian level

As we mentioned in the Introduction the phase factor in ordinary Niederer's transformation (1.4) appeared exactly under the total time derivative in the formula joining the harmonic oscillator Lagrangian with the free one (see (1.3)); therefore, we expect the same holds true for the PU model. However, here the situation is slightly more complicated since we have various Lagrangian and Hamiltonian formalisms, e.g. the phase factor g cannot occur on the Lagrangian level since the Hamiltonian H_3 has no clear Lagrangian formulation (see, [47]). On the other hand, analyzing the Ostrogradsky method one can see that, in analogy to the ordinary mechanics, the modification of the Lagrangian consisting in adding the total time derivative of a certain function added to the Lagrangian shifts the Ostrogradsky momenta by partial derivatives of this function with respect to the consecutive time derivatives of the coordinate. Next, replacing these time derivatives by the Ostrogradsky coordinates we obtain a canonical transformation related to this modification of the Lagrangian. If we also change the coordinate in the Lagrangian (as it is in the PU case) the transformation rule for momenta becomes slightly more complicated. However, it is still possible to find the total time derivative provided we know the canonical transformation (generating function) between the Hamiltonians. Thus, in the case of the Lagrangians (1.5) and (1.9), we can deduce that the following identity holds:

$$Ldt \equiv \tilde{L}d\tilde{t} - dh(\tilde{q},\tilde{t}); \tag{4.1}$$

equivalently

$$\frac{1}{2}\sum_{k=0}^{n}(-1)^{k-1}\sigma_k\left(\frac{d^kQ}{dt^k}\right)^2 = \frac{1}{2}\tilde{\kappa}^2\left(\frac{d^n\tilde{q}}{d\tilde{t}^n}\right)^2 - \tilde{\kappa}^2\frac{dh(\tilde{q},\tilde{t})}{d\tilde{t}}; \tag{4.2}$$

where $h(\tilde{q},\tilde{t})$ is the phase factor $h(\tilde{q}_0,\dots,\tilde{q}_{n-1},\tilde{t})$ of the unitary transformation $\hat{W}$ after substituting

$$\tilde{q}_k = \tilde{q}^{(k)} \equiv \frac{d^k\tilde{q}}{d\tilde{t}^k}; \tag{4.3}$$

while on the left hand side of eq. (4.2) Q and its derivatives are expressed in terms of $\tilde{q}$ (by virtue of (1.11)). Due to (3.13) and (3.16) we have

$$\begin{aligned} h(\tilde{q}_0,\dots,\tilde{q}_{n-1}) = {} & f(q_0(\tilde{q}_0,\dots,\tilde{q}_{n-1},\tilde{t}),\dots,q_{n-1}(\tilde{q}_0,\dots,\tilde{q}_{n-1},\tilde{t}))+ \\ & g(\tilde{q}_0,\dots,\tilde{q}_{n-1},\tilde{t}), \end{aligned} \tag{4.4}$$

where $q_k = q_k(\tilde{q}_0,\dots,\tilde{q}_{n-1},\tilde{t})$ are given by eq. (3.2). Of course, the relation (4.2) can be checked explicitly; however, the computations are rather long and we only give the main idea. First, we note that the compositions of the transformation (2.16) and (3.2) give $\frac{dQ_k}{dt} = Q_{k+1}$, $k = 1,\dots,n-1$, provided the identity (4.3) holds. This gives the time derivatives $\frac{d^kQ}{dt^k}$ in terms of $\tilde{q}$ and, consequently, enables us to express the left hand side of (4.2) in terms of $\tilde{q}$. Next, using (3.14) and (3.15) one can compute the total time derivative of g. The most complicated point is to compute the time derivative of f; to this end the relations (A.2)–(A.9) appear to be helpful.

We conclude taking $n = 2$ as an example. In this case one finds

$$Q_1 = Q = -\tilde{\kappa}^{-3}\tilde{q}, \quad Q_2 = \frac{dQ}{dt} = 3\tilde{\kappa}^{-2}\dot{\tilde{\kappa}}\tilde{q} - \tilde{\kappa}^{-1}\dot{\tilde{q}}, \tag{4.5}$$

and

$$h(\tilde{q},\tilde{t}) = 3\tilde{\kappa}^{-3}\dot{\tilde{\kappa}}(3\omega^2\tilde{\kappa}^{-2} - 2\dot{\tilde{\kappa}}^2)\tilde{q}^2 + 3\tilde{\kappa}^{-2}(-\omega^2\tilde{\kappa}^{-2} + 2\dot{\tilde{\kappa}}^2)\tilde{q}\dot{\tilde{q}} - 2\tilde{\kappa}^{-1}\dot{\tilde{\kappa}}\dot{\tilde{q}}^2. \tag{4.6}$$

Substituting (4.5) into the left hand side of eq. (4.2) and (4.6) into the right hand side we obtain the desired identity.

5. Conclusions

In this paper we have completed the picture drawn in Refs. [41,47] by showing that generalized Niederer's transformation which relates the free higher derivatives theory to the Pais–Uhlenbeck oscillator can be constructed also in the quantum domain. We obtained an unitary operator which maps the solutions of the Schrödinger equation for the PU oscillator (with odd frequencies) into the solutions of the Schrödinger equation corresponding to the free higher derivatives theory. Moreover, we showed that, in the case of the Schrödinger equation with the Ostrogradsky Hamiltonian, the phase factor entering this operator enters also as the total time derivative joining the nondegenerate Lagrangian (1.9) with the free one (1.5). These results lead to the conclusion that the maximal (kinematical) symmetry algebra of the quantum PU model with the odd frequencies is isomorphic to the central extension of $l = n - \frac{1}{2}$ conformal Galilei algebra. However, it is an interesting question whether for arbitrary frequencies the group of quantum symmetries is broken to a simpler one (without the conformal and dilatation transformations). Turning to further developments, let us note that the quantum transformation obtained here can be used in various ways as it is in the case for ordinary Niederer's transformation. For example, it can help to find the Feynman propagator for the general PU model (which is a rather complicated task even in the case of $n = 2$, if we use the standard methods cf. [23,27]).

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Appendix A

This Appendix contains some relations which are crucial for the main part of the paper. Following Ref. [46] we introduce the Fourier expansion coefficients $\gamma^{\pm}_{kp}$

$$\sin^p t \cos^{2n-1-p} t = \begin{cases} \sum_{k=1}^{n} \gamma^{+}_{kp} \cos(2k-1)t, & p = 0, \dots, 2n-1, \text{ even;} \\ \sum_{k=1}^{n} \gamma^{-}_{kp} \sin(2k-1)t, & p = 0, \dots, 2n-1, \text{ odd;} \end{cases} \tag{A.1}$$

Denoting by $\beta^{\pm}$ the inverse matrix of $\gamma^{\pm}$ and putting, by definition, $\gamma^{\pm}_{kp} = 0$ whenever $p < 0$, $p > 2n-1$, $k < 1$, $k > n$ we have the following relations

$$\gamma^{+}_{kp} = (-1)^{k-1} \gamma^{-}_{k,2n-1-p}, \qquad \beta^{+}_{pk} = (-1)^{k-1} \beta^{-}_{2n-1-p,k}, \tag{A.2}$$

$$\beta^{\pm}_{pk} = \frac{4^{n-1}(n-k)!(n+k-1)!}{p!(2n-1-p)!} \gamma^{\pm}_{kp}, \tag{A.3}$$

$$(2k-1)\gamma^{\pm}_{kp} = \mp p \gamma^{\mp}_{k,p-1} \pm (2n-1-p)\gamma^{\mp}_{k,p+1}, \tag{A.4}$$

for $k = 1, \dots, n$ and $p = 0, \dots, 2n-1$. Moreover, in the odd case ($\omega_k = (2k-1)\omega$, for $k = 1, \dots, n$) we have

$$\sum_{k=1}^{n} \gamma_{km}^{+} \omega_k^{r-1} = 0; \quad m > n-1, \quad m\text{-even}, r = 1, \ldots, n, \quad r\text{-odd}, \tag{A.5}$$

$$\sum_{k=1}^{n} \gamma_{km}^{-} \omega_k^{r-1} = 0; \quad m > n-1, \quad m\text{-odd}, r = 1, \ldots, n, \quad r\text{-even}, \tag{A.6}$$

$$\sum_{k=1}^{n} \gamma_{k,n-1}^{\mp} \omega_k^{n-1} = (n-1)!(-1)^{[\frac{n-1}{2}]} \omega^{n-1}, \quad +(-) \text{ for } n\text{-odd(even)}; \tag{A.7}$$

$$\sum_{j=0}^{n} \sigma_j (-1)^{j-1} \sum_{k,\bar{k}=1}^{n} \rho_k \gamma_{\bar{k},\bar{m}}^{\pm} \beta_{mk}^{\pm} {\sum_{r=1}^{j}}' {}^{('')} \omega_k^{2j-r-1} \omega_{\bar{k}}^{r-1} = \delta_{m\bar{m}}, \tag{A.8}$$

where $m, \bar{m} = 0, \ldots, n-1$: in the case $m, \bar{m}$-even we take "+" sign and one prime while in the case $m, \bar{m}$-odd we take "−" sign and double prime; also

$$\frac{\omega^{-n+1}}{(n-1)!} \sum_{j=0}^{n} \sigma_j (-1)^j \sum_{k,\bar{k}=1}^{n} (-1)^{k-1} \gamma_{k,n}^{\pm} \gamma_{\bar{k}m}^{\pm} {\sum_{r=1}^{j}}' {}^{('')} \omega_k^{2j-r} \omega_{\bar{k}}^{r-1} = (-1)^{[\frac{n}{2}]} \sum_{k=1}^{n} \gamma_{km}^{\pm} \omega_k^{n}, \tag{A.9}$$

where $m = 0, \ldots, n-1$: in the case m, n-even "+" sign and one prime, in the case m, n-odd "−" sign and double prime have to be chosen.

The identities (A.2)–(A.4) can be found in Ref. [46]. The relations (A.5)–(A.7) are obtained by differentiating repeatedly (A.1) at $t = 0$. Finally, eqs. (A.8)–(A.9) follow from (A.2)–(A.7) after some calculations.

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